

DISTORTION OF LEAVES IN PRODUCT FOLIATIONS

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ABSTRACT. We produce examples of codimension one foliations of \mathbb{H}^2 and \mathbb{E}^2 with bounded geometry which are topologically product foliations, but for which leaves are *non-recursively distorted*. That is, the function which compares intrinsic distances in leaves with extrinsic distances in the ambient space grows faster than any recursive function.

1. INTRODUCTION

It is a basic problem, given a foliation of a Riemannian manifold, to compare the extrinsic and the intrinsic geometry of leaves of the foliation. For leaves which are compact, this amounts to studying the distortion of π_1 of the leaf in π_1 of the ambient manifold; much is known about this problem. For a partial survey, see [4].

For taut codimension one foliations of 3-manifolds, one knows by a result of Sullivan in [5] that one can choose a metric on the ambient manifold such that the leaves are minimal surfaces. One could think of this as saying that taut foliations “measure area well”. However, leaves of taut foliations are far from being quasi-isometrically embedded, in general. For instance, a theorem of Fenley in [2] states that for an \mathbb{R} -covered foliation \mathcal{F} of a hyperbolic 3-manifold M , leaves of the pulled-back foliation $\tilde{\mathcal{F}}$ of \tilde{M} limit to the entire sphere at infinity of $\tilde{M} = \mathbb{H}^3$. A foliation is \mathbb{R} -covered if the pulled-back foliation $\tilde{\mathcal{F}}$ is topologically the product foliation of \mathbb{R}^3 by horizontal \mathbb{R}^2 's. Fenley's proof is attractive but somewhat lengthy, and therefore we think it is useful to give a simple proof of this fact here:

Theorem 1.1. *For \mathcal{F} an \mathbb{R} -covered foliation of M , a finite volume hyperbolic 3-manifold, every leaf λ of $\tilde{\mathcal{F}}$ limits to the entire sphere at infinity S_∞^2 of $\tilde{M} = \mathbb{H}^3$.*

Proof: For a leaf λ of $\tilde{\mathcal{F}}$, let $\lambda_\infty = \bar{\lambda} - \lambda \subset S_\infty^2$ be the set of points in S_∞^2 in the closure of λ . We claim that $\lambda_\infty = S_\infty^2$ for each λ .

Suppose some λ_∞ omits some point $p \in S_\infty^2$, and therefore an open disk U containing p . Let U' and U'' be two open disks in S_∞^2 whose closures are disjoint and contained in U . Since M has finite volume, there are elements $\alpha, \beta \in \pi_1(M)$ such that $\alpha(S_\infty^2 - U) \subset U'$ and $\beta(S_\infty^2 - U) \subset U''$. Let $\lambda' = \alpha(\lambda)$ and $\lambda'' = \beta(\lambda)$. Since the leaf space of $\tilde{\mathcal{F}}$ is \mathbb{R} , After relabeling the leaves if necessary, we can assume λ separates λ' from λ'' . However, there is clearly an arc γ in S_∞^2 joining λ'_∞ to λ''_∞ which avoids λ_∞ . γ is a Hausdorff limit (in $\mathbb{H}^3 \cup S_\infty^2$) of arcs $\gamma_i \subset \mathbb{H}^3$ running between λ' and λ'' . Each γ_i must intersect λ in some point q_i , and by extracting a subsequence, we find $q_i \rightarrow q \in \gamma$. But by construction, $q \in \lambda_\infty$, giving us a contradiction. \square

If we want to study the distortion of leaves in taut foliations, then \mathbb{R} -covered foliations are the most delicate case. For, if \mathcal{F} is a taut foliation which is *not* \mathbb{R} -covered, then the leaf space of $\tilde{\mathcal{F}}$ is non-Hausdorff. That is, there are a sequence of pairs of points $p_i, q_i \in \tilde{M}$ with $p_i \rightarrow p$ and $q_i \rightarrow q$ such that $p_i, q_i \in \lambda_i$ but p and q lie on *different* leaves of $\tilde{\mathcal{F}}$. It follows that the distance between p_i and q_i *as measured in \mathcal{F}* is going to infinity; that is, $d_{\lambda_i}(p_i, q_i) \rightarrow \infty$. On the other hand, the distance between p_i and q_i *as measured in \tilde{M}* is bounded: $d_{\tilde{M}}(p_i, q_i) \rightarrow d_{\tilde{M}}(p, q)$.

By contrast, leaves of \mathbb{R} -covered foliations are never infinitely distorted. In fact in [1] we show the following:

Theorem 1.2. *Let \mathcal{F} be an \mathbb{R} covered foliation of an atoroidal 3-manifold M^3 . Then leaves λ of $\tilde{\mathcal{F}}$ are quasi-isometrically embedded in their t -neighborhoods, for any t . That is, there is a uniform K_t, ϵ_t such that for any leaf λ of $\tilde{\mathcal{F}}$, the embedding $\lambda \rightarrow N_t(\lambda)$ is a (K_t, ϵ_t) quasi-isometry.*

A heuristic scheme to measure the distortion of a leaf λ runs as follows.

We assume that \mathcal{F} is co-oriented, so that in the universal cover there is a well-defined notion of the space “above” and the space “below” a given leaf. Since leaves are minimal surfaces, the mean curvature is zero, so there will be well-defined approximate directions at every point in which the leaf “bends upwards” and approximate directions in which it “bends downwards”. For an arc α between points p, q of a leaf λ which is roughly parallel to a direction of positive extrinsic curvature, the geodesic α^+ in \tilde{M} running between p and q will lie mostly above λ . We can cap off the circle $\alpha \cup \alpha^+$ with a disk D_α of minimal area, and consider its projection to M . The intersection of D_α with $\tilde{\mathcal{F}}$ gives a foliation of D_α by arcs parallel to the principal directions of positive curvature in the leaves of $\tilde{\mathcal{F}}$. Thus the projection of D_α to M should be close to an embedding, since its self-intersections cannot have a large dihedral angle. If we consider longer and longer leafwise geodesics α , the geometric limit of the disks D_α in M should be a geodesic lamination Λ^+ in M which describes the “eigendirections” of positive curvature of \mathcal{F} . If we consider the principal directions of negative curvature, we should get a complementary lamination Λ^- , transverse to \mathcal{F} and to Λ^+ , which describes the “eigendirections” of negative curvature of \mathcal{F} .

To study the distortion of leaves of \mathcal{F} in such a setup, one need only study, for a typical leaf l of Λ^+ say, how distorted the induced one-dimensional foliation $\mathcal{F} \cap l$ is in l . Such a foliation has *bounded geometry* — its extrinsic curvature is bounded above and below by some constant — because of the compactness of M . Moreover it is topologically a product. The point of this paper is to show that these two conditions in no way allow one to establish any bound on the distortion function of \mathcal{F} .

One remark worth making is that a pair of laminations Λ^\pm transverse to an \mathbb{R} -covered foliation of an atoroidal 3-manifold M are constructed in [1] and independently by Sérgio Fenley ([3]). The interpretation of these foliations as eigendirections of extrinsic curvature is still conjectural, however.

2. NON-RECURSIVE DISTORTION

Definition 2.1. The *distortion function* \mathcal{D} of a foliation \mathcal{F} in a Riemannian manifold M is defined as follows: if $d_{\mathcal{F}}$ denotes the intrinsic distance function in a leaf of \mathcal{F} thought of as a geodesic metric space, and d_{amb} denotes the Riemannian distance

function in the ambient space, then

$$\mathcal{D}(t) = \sup_{x, y \in \lambda | d_{\text{amb}}(x, y) = t} d_{\mathcal{F}}(x, y)$$

where the supremum is taken over all pairs of points in all leaves of \mathcal{F} . We usually expect that M is simply connected.

Theorem 2.1. *Let λ be a smooth oriented bi-infinite ray properly immersed in \mathbb{H}^2 .*

1. *If $-1 \leq \kappa \leq 1$ everywhere, then the distortion function \mathcal{D} is at most exponential.*
2. *If $\kappa \geq 1$ everywhere and > 1 somewhere, then λ has a self-intersection.*
3. *If $\kappa = 1$ everywhere then λ is a horocircle and therefore has exponential distortion.*

Proof: The proof of 1. is a standard comparison argument, and amounts to showing that λ makes at least as much progress as a horocircle.

To see that 2. holds, we consider the progress of the *osculating horocircle* to λ . At a point x on λ , the osculating horocircle H_x is the unique horocircle on the positive side of λ which is tangent to λ at x . Let $B_x \in S_\infty^1$ be the basepoint of H_x . Then the curvature condition on λ implies that as one moves along λ in the positive direction, B_x always moves anticlockwise. Since $\kappa > 1$ somewhere, the derivative of B_x is nonzero somewhere. If we truncate λ on some sufficiently big compact piece and fill in the remaining segments with horocircles, we get a properly immersed arc λ' in B^2 with positive winding number, which consequently has a self-intersection somewhere. If this intersection is in the piece agreeing with λ , we are done. Otherwise, the curvature condition implies that λ must have an intersection in the region enclosed by λ' . \square

In light of this theorem, it is perhaps surprising that we can make the following construction:

Theorem 2.2. *For any $\epsilon > 0$ there is a foliation \mathcal{F} of \mathbb{H}^2 which is topologically the standard foliation of \mathbb{R}^2 by horizontal lines with the following properties:*

- *Leaves are smooth.*
- *The extrinsic curvature of any leaf at any point is bounded between $1 - \epsilon$ and $1 + \epsilon$.*
- *The distortion function \mathcal{D} grows faster than any recursive function.*

Proof: We consider the upper half-space model of \mathbb{H}^2 and let λ be the leaf passing through the point i . λ will be the graph of a function $r = \phi(\theta)$ in polar co-ordinates for $\theta \in (0, \pi)$. Choose some very small δ . Then for $\pi - \delta > \theta > \delta$ we let λ agree with the horocircle with “center” at ∞ passing through i . Let r_n be a sequence of positive real numbers which grows faster than any recursive function. Then we define

$$\phi\left(\frac{\delta}{2^n}\right) = r_n$$

Since r_n grows so very fast, one can easily choose ϕ to interpolate between $\frac{\delta}{2^n}$ and $\frac{\delta}{2^{n+1}}$ so that λ is very close to a radial (Euclidean) line of very small slope in this range. The extrinsic curvature of such a line in \mathbb{H}^2 is very close to 1. Define ϕ in the range $\theta > \pi - \delta$ by $\phi(\theta) = \phi(\pi - \theta)$. Then the extrinsic distance between (r, θ) and

$(r, \pi - \theta)$ depends only on θ . It follows that for θ between $\frac{\delta}{2^n}$ and $\frac{\delta}{2^{n+1}}$ the extrinsic distance between (r, θ) and $(r, \pi - \theta)$ is bounded above by some recursive function in n . However, the length of the path in λ between (r, θ) and $(r, \pi - \theta)$ is growing approximately like $\ln(r_n)$. The distortion function \mathcal{D} is therefore non-recursive.

Let $\lambda_1 = \lambda$ and for $t > 0$, define λ_t to be the graph of the function $r = t\phi(\theta)$ in polar co-ordinates. Then there is a hyperbolic isometry taking λ_t to λ_s for any s, t and therefore each of the λ_t satisfies the same curvature bounds as λ . Moreover, the union of the λ_t gives a product foliation of \mathbb{H}^2 , as required. \square

Remark 2.1. For those unfamiliar with the concept, it should be noted that it is easy to produce a function (even an integer valued function) such as r_n which grows faster than any recursive function. For, enumerate all the recursive functions somehow as ϕ_n . Then define $r(n) = \max_{m \leq n} \phi_m(n)$. Such an r grows (eventually) at least as fast as any recursive function, and therefore faster than any recursive function.

Essentially the same construction works for Euclidean space, a fact which we now establish.

Theorem 2.3. *For any $\epsilon > 0$ there is a foliation \mathcal{F} of \mathbb{E}^2 which is topologically the standard foliation of \mathbb{R}^2 by horizontal lines with the following properties:*

- *Leaves are smooth.*
- *The extrinsic curvature of any leaf at any point is bounded by ϵ*
- *The distortion function \mathcal{D} grows faster than any recursive function.*

Proof: The leaf λ of \mathcal{F} passing through the origin will be the graph of an even function $y = \phi(x)$ and every other leaf will be a translate of λ . After choosing a sufficiently large K and small δ , we make $\phi(x) = \delta x^2$ for $|x| < K$, so that λ has curvature bounded below ϵ , and is very nearly vertical at $x = K$. Then let r_i be a sequence growing faster than any recursive function, as before, and define $\phi(K+n) = r_n$. Then we can choose ϕ to interpolate between $K+n$ and $K+n+1$ for each n to make it very smooth and almost straight, since λ will be almost vertical in this region. \square

3. MINIMALITY

Definition 3.1. A *decorated* metric space is a metric space in the usual sense with some auxiliary structure. For instance, this structure could consist of a basepoint, some collection of submanifolds, a foliation or lamination, etc. together with a notion of convergence of such structures in the geometric topology, in the sense of Gromov. One says a decorated metric space M has *bounded geometry* if the metric space itself has bounded geometry, and if for every t , the decorated metric spaces obtained as restrictions of M to the balls of radius t form a precompact family.

For instance, we might have a family of foliated Riemannian manifolds M_i, \mathcal{F}_i . The geometric topology requires a choice of basepoints p_i in each M_i . Then we say the family $(M_i, \mathcal{F}_i, p_i)$ is a Cauchy sequence if there are a sequence of radii r_i and $\epsilon_i \rightarrow 0$ so that the Gromov–Hausdorff distance between the ball of radius r_i in M_i and M_j about p_i, p_j for $j > i$ is at most ϵ_i , and that such “near-isometries” can be chosen in a way that the leaves of \mathcal{F}_i can be taken ϵ_i -close to the leaves of \mathcal{F}_j . From such a Cauchy sequence we can extract a limit (M, \mathcal{F}, p) .

Definition 3.2. A decorated metric space M with bounded geometry is *minimal* if for every limit (M^+, p) of the pointed decorated metric spaces (M, p_i) and every point $q \in M$, there is a sequence $q_i \in M^+$ such that the pointed decorated metric spaces $(M^+, q_i) \rightarrow (M, q)$.

The foliations constructed in the last section are certainly not minimal. We ask the question: does the assumption of minimality allow one to make some estimate on the distortion function for a product foliation?

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